University of Michigan — 上海交通大学交大密西根 联合学院 UM–SJTU Joint Institute — Shanghai Jiao Tong University

Dr. Horst Hohberger — Summer Semester '07

Exercise Set 1 for Calculus 186

Date Due: 10:00 AM, Monday, the 9th of April 2007

This exercise sheet is longer than the average exercise sheets will be in this semester. The reason is that I wish to obtain an impression of the level of your mathematical knowledge and skills. Thus, this sheet contains some easy and some very difficult problems. As with many problems in real life, you are not told in advance which are simple and which are more involved, but are supposed to find out for yourself. The problems should require only knowledge of the techniques taught in Calculus 185, and of mathematical induction, which is the subject of the first lecture in this semester.
I want to take the opportunity to set some ground rules for the completion of the exercises for this course:

i) There will be an examination at the end of the course. In order to qualify for the examination, you will need to obtain 50% of the total number of marks of all Exercise Sets.

ii) You are required to *hand in the exercises on time.* No exercises will be accepted after the due date.

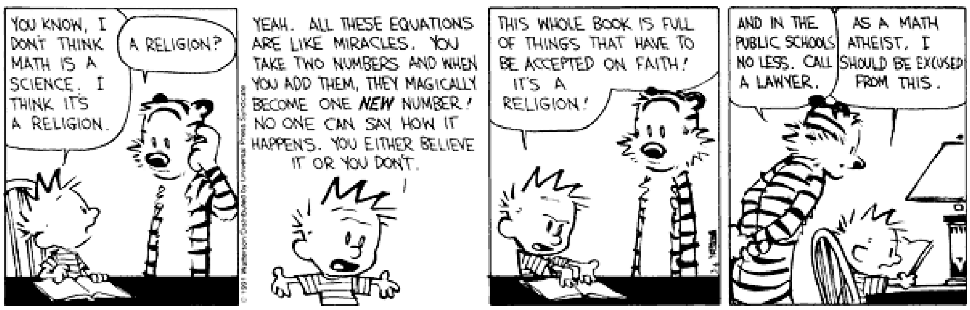
iii) You are required to compose your solutions in *neat and legible handwriting.* The Teaching Assistants have been instructed to deduct up to 10% of the total score for messiness. Unreadable handwriting will not be graded.

iv) In order to obtain the highest possible score, make sure that you explain your reasoning. Often, simple formulae are not enough to answer a question. *Explain what you are doing!* This will also ensure that you get a large fraction of the total points even if you make a mistake in your calculations

v) You are encouraged to cooperate with other students. Feel free to discuss problenms and develop solutions in groups. But *you are not allowed to simply copy other students' work!* If the Teaching Assistants suspect that you have copied someone else's answers, no points will be awarded for the entire Exercise Set!

vi) If you have any problems, questions or comments regarding the exercises, the lecture or mathematics, please *visit me during my office hours.* This time has been set aside specifically for you, and you should make as much use fo it as you can. If you wish for comments to reach me anonymously, please talk to the Teaching Assistants.

Office hours for Calculus 186: Mondays, 12:30-13:30 and Fridays, 12:30-13:30



Exercise 1. Find the solution set of the inequality $|x+2| \le |x-1|$, $x \in \mathbb{R}$.
(4 Marks)

Exercise 2. The following two results are extensions of the binomial forumla:

i) Use mathematical induction to show the *Leibniz rule* for the nth derivative of the product of two functions:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

ii) Prove the *multinomial expansion*

$$(x_1 + \cdots + x_k)^n = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1! \cdots n_k!} x_1^{n_1} \cdots x_k^{n_k},$$

where $k \in \mathbb{N} \setminus \{0\}$ and $n_1, \ldots, n_k \in \mathbb{N}$.

(2 + 3 Marks)

Exercise 3. Prove the following identities of binomial copefficients.

i) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad n \in \mathbb{N} \setminus \{0\}.$

ii) $\displaystyle\sum_{j=0}^{n-k} \binom{k+j}{k} = \binom{n+1}{k+1}, \quad k \in \mathbb{N},\ n \ge k.$

iii) $\displaystyle\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}, \quad n \in \mathbb{N} \setminus \{0\}.$

iv) $\displaystyle\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}, \quad n \in \mathbb{N} \setminus \{0,1\}.$

v) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \quad n \in \mathbb{N}.$ (This is known as *Vandermonde's convolution formula*; a more general formula was known to the Chinese mathematician Zhu Shijie (朱世杰, also known as Chu Shih-Chieh), who lived in the 13th century. An abstract generalization of this formula is known as the Chu-Vandermonde formula.)

(1 + 2 + 1 + 2 + 3 Marks)

Exercise 4. Explain in your own words what the words "limit of a function", "continuous function" and "differentiable function" mean.
(3 × 1 Marks)

Exercise 5. For fixed $a, b, c \in \mathbb{R}$, find $\alpha, \beta \in \mathbb{R}$, such that

$$\lim_{x\to\infty} \sqrt{ax^2 + bx + c} - \alpha x - \beta = 0.$$

Having found such $\alpha, \beta \in \mathbb{R}$, can there exist different numbers $\alpha', \beta' \in \mathbb{R}$ such that $\lim_{x\to\infty} \sqrt{ax^2+bx+c} - \alpha' x - \beta' = 0$? Explain!
(2 + 2 Marks)

Exercise 6. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function with image contained in $[a,b]$, i.e., $f([a,b]) \subset [a,b]$. Show that f then has a fixed point, i.e., there exists an $x_0 \in [a,b]$ such that $f(x_0) = x_0$
(2 Marks)

Exercise 7.

i) Use the Mean Value Theorem to prove the following statement: Let $I = [x_0, x_1] \subset \mathbb{R}$ or $I = [x_0, \infty)$ and $f, g\colon I \to \mathbb{R}$ be differentiable on the interior of I. Assume that $f(x_0) \le g(x_0)$ and $f' \le g'$ on I. Then $f \le g$ on I.

ii) Show the following inequalities for $x > 0$:

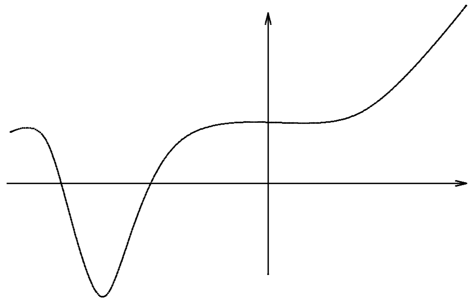
$$x - \frac{x^3}{6} \le \sin x \le x, \qquad\qquad 1 - \frac{x^2}{2} \le \cos x.$$

(2 + 3 Marks)

Exercise 8. Calculate the 100th derivative of the real functions $(x^2 + 3x + 2)^{-1}$ and $\dfrac{x^2+1}{x^3-x}$.

(1 + 1 Marks)

Exercise 9. The function $f\colon \mathbb{R} \to \mathbb{R}$ has the following graph on $G \subset \mathbb{R}$, $0 \in G$:



Sketch the graph of the derivative $f'(x)$ and the integral $\int_0^x f$ for $x \in G$.

(2 + 2 Marks)

Exercise 10. Calculate the following integrals:

$$\text{i) } \int_0^1 \ln x\, dx, \quad \text{ii) } \int \frac{\ln\ln x}{x}\, dx, \quad \text{iii) } \int e^x \sin x\, dx, \quad \text{iv) } \int_1^{e^{\pi/2}} \sin\ln x\, dx, \quad \text{v) } \int_0^{\frac{\pi}{2}} \ln\sin x\, dx,$$

$$\text{vi) } \int \tan x\, dx, \quad \text{vii) } \int \tan^2 x\, dx, \quad \text{viii) } \int \frac{1}{x^2(1+x)^2}\, dx, \quad \text{xi) } \int e^{x^2} x(1+x^2)\, dx, \quad \text{x) } \int \sqrt{\frac{1-x}{1+x}}\, dx.$$

(10 × 1 Mark)

Exercise 11. Show the following estimates:

$$\int_1^3 \sqrt{x^4+1}\, dx \ge 26/3 \qquad \int_0^{\pi/2} x \sin x\, dx \le \pi^2/8 \qquad \left| \int_0^{\pi} x^2 \cos x\, dx \right| \le \pi^3/3$$

(3 × 2 Marks)

Mathematician of the Week - Zhu Shijie

Zhu Shijie (Chinese: 朱世杰, Styled Hanqing 字漢卿，號松庭) (mid-1270s?-1330?) also known as Chu Shih-Chieh was one of the greatest Chinese mathematicians.
Little is known about his life, but two of his mathematical works have survived. Introduction to Mathematical Studies 《算學啟蒙》, written in 1299, is an elementary textbook on mathematics. Zhu included about 260 problems to explain operations in arithmetic and algebra. This book also showed how to measure different two-dimensional shapes and three-dimensional solids. The Introduction had an important influence on the development of mathematics in Japan. The book was once lost in China until a copy of the book was made from a Korean source in 1839.
Zhu's second book, Precious Mirror of the Four Elements 《四元玉鑒》(1303), is his most important work. With this book, Zhu brought Chinese algebra to its highest level. It includes an explanation of his method of the four elements, which are used to signify four unknown quantities in a single algebraic equation. Zhu also explained how to find square roots and added to the understanding of series and progressions. The preface of the book describes how Zhu traveled around China for 20 years as a teacher of mathematics.
Read more about Zhu Shijie at

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Zhu_Shijie.html`